Full counting statistics of chaotic cavities with many open channels

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Explicit formulas are obtained for all moments and for all cumulants of the electric current through a quantum chaotic cavity attached to two ideal leads, thus providing the full counting statistics for this type of system. The approach is based on random matrix theory, and is valid in the limit when both leads have many open channels. For an arbitrary number of open channels we present the third cumulant and an example of non-linear statistics.

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The physics of current fluctuations in mesoscopic conductors is an interesting and fundamental quantum mechanical problem, since at low temperatures they are mainly due to the discreteness of the electron charge. The study of shot noise, for example, is an active area of theoretical and experimental research involving different types of systems (e.g. quantum dots, disordered wires, quantum point contacts) and different regimes[1] (e.g. Coulomb blockade, quantum Hall effect, localization). A relatively recent approach is the concept of full counting statistics[2], the study of all cumulants of the charge fluctuation, which amounts to having complete information concerning charge counting in a transport process. This approach has recently attracted much attention and has been applied in a wide variety of situations (see Ref. 3 and references therein). Experimental measurements of the third moment of these fluctuations have already been reported[4].

In the case of chaotic cavities the random matrix theory (RMT) approach[5] has been very successful in reproducing different experimental observations related to quantum transport, such as weak localization and universal conductance fluctuations. If the cavity is connected to two ideal leads, supporting respectively N_1 and N_2 open channels, the conductance is given by the Landauer-Büttiker formula $g = G_0\mathrm{Tr}[T]$, where G_0 is the conductance quantum, $T = t^\dagger t$ and t is the transmission matrix. However, only very recently was an exact expression obtained[6] within RMT for the shot noise $P = P_0\mathrm{Tr}[T(1-T)]$ (with $P_0 = 2eVG_0$ where V is a small voltage bias) and an explicit general result is not available for higher moments of the type $\mathrm{Tr}[T^m]$. For chaotic cavities with large N_1, N_2 the third and fourth cumulants of charge transfer have been obtained[7], as well as an expression for the cumulant generating function[8].

On the other hand, recent semiclassical calculations based on correlated classical trajectories that transmit through the cavity have been able to reproduce the RMT results, both for the conductance and for the shot noise. These calculations have a natural perturbative structure on the parameter N^{-1}, where $N = N_1 + N_2$ is the total number of channels. Initially the leading order expressions were reproduced,[9,10] later the full series were obtained and exactly summed.[11] The next natural step would be to tackle $\mathrm{Tr}[T^m]$, and it is thus of interest to have the corresponding RMT prediction of this quantity, at least to leading-order in N^{-1}. This is the purpose of this work.

We will be interested in the dimensionless moments defined as $\sum_{i=1}^n T_i^m$, where T_i are the eigenvalues of the matrix T and $n = \min\{N_1, N_2\}$. Within RMT the T_i are correlated random numbers between 0 and 1, whose distribution depends only on the symmetries of the system (orthogonal, unitary or symplectic, labeled by $\beta = 1, 2$ or 4 respectively). The average value of the moments are then given simply by

$$M_m = n\langle T_1^m\rangle. \tag{1}$$

The distribution of transmission eigenvalues can be characterized by a density, $\rho_\beta(T)$, such that $\langle T_1^m\rangle = \int_0^1 \rho_\beta(T)T^m dT$, or equivalently by a joint probability distribution $\mathcal{P}_\beta$ such that

$$\langle T_1^m\rangle = \int_0^1 dT_1 \cdots \int_0^1 dT_n \; T_1^m \mathcal{P}_\beta(T). \tag{2}$$

The expression for $\mathcal{P}_\beta(T)$ is[5]

$$\mathcal{P}_\beta(T) = \mathcal{N}_\beta^{-1}|\Delta(T)|^\beta \prod_{j=1}^n T_j^\alpha, \tag{3}$$

where $\Delta(T) = \prod_{i<j}(T_i - T_j)$ is the Vandermonde determinant, $\alpha = \frac{\beta}{2}(|N_2 - N_1| + 1) - 1$ and $\mathcal{N}_\beta$ is a normalization constant. In Ref. 6 the authors used simple recurrence relations from the theory of Selberg's integral[12] to obtain an exact result with arbitrary N_1, N_2 for the second moment M_2 and for the shot-noise (second cumulant). Here we follow a similar approach and compute the third cumulant, sometimes called the skewness. Moreover, we then proceed to obtain explicit formulas for all moments M_m and for all cumulants, valid to first order in the inverse number of channels, i.e. in the limit $N_1, N_2 \gg 1$.

We must note that in the semiclassical limit of short wavelengths some noiseless scattering states can be created,[13] leading to a breakdown of the universality implied by RMT predictions.[14,15] This phenomenon is governed by the ratio τ_E/τ_D of the quantum Ehrenfest time to the classical dwell time, and its influence has

been investigated on shot-noise,[10,16] the weak localization effect[17] and conductance fluctuations.[14,18] Our results are restricted to the universal regime $\tau_E/\tau_D \to 0$, when these system-specific corrections are neglected.

Let us consider a certain fixed sequence of k positive integers, $\mathbf{m} = [m_1, ..., m_k]$, and for any subsequence of length $q \le k$ let us define the function

$$P^q_{\mathbf{m}}(T) = \prod_{j=1}^{q} T_j^{m_j}, \quad P^0_{\mathbf{m}}(T) = 1. \tag{4}$$

We take now $T_k P^k_{\mathbf{m}}(T)\mathcal{P}_\beta(T)$, derive it with respect to T_k and integrate over all variables to obtain

$$F = (\alpha + m_k)\langle P^k_{\mathbf{m}}(T)\rangle + \beta \sum_{j=2}^{n} \left\langle P^k_{\mathbf{m}}(T) \frac{T_k}{T_k - T_j} \right\rangle, \tag{5}$$

where the constant F is given by

$$F = \int_0^1 dT_1 \cdots \int_0^1 dT_n \frac{d}{dT_k}\left[T_k P^k_{\mathbf{m}}(T)\mathcal{P}_\beta(T)\right]. \tag{6}$$

We can see that F is actually independent of m_k. Hence, we may equate the r.h.s. of (5) at different values of this variable arriving at a recurrence relation. To solve this relation in general is presently beyond reach, but armed with some patience once can compute the first moments. This is essentially what was done in Ref. 6. We take it a bit further and find the third moment. Instead of writing the lengthy expression that arises for M_3 we present the corresponding cumulant (assuming for simplicity $N_2 \ge N_1$),

$$\frac{Q_3}{Q_2} = -\frac{(N_2 - N_1 + 1 - \frac{2}{\beta})(N_2 - N_1 - 1 + \frac{2}{\beta})}{(N - 1 + \frac{6}{\beta})(N - 3 + \frac{2}{\beta})}, \tag{7}$$

where Q_2 is the average shot noise in units of P_0,

$$\frac{\langle P\rangle}{P_0} = Q_2 = \frac{N_1 N_2 (N_1 - 1 + \frac{2}{\beta})(N_2 - 1 + \frac{2}{\beta})}{(N - 1 + \frac{2}{\beta})(N - 2 + \frac{2}{\beta})(N - 1 + \frac{4}{\beta})}. \tag{8}$$

The result for Q_3 agrees in the limit $N \gg 1$ with the one presented in Ref. 7.

It is also possible to go beyond linear statistics, and compute higher correlations as for example

$$\begin{aligned}&\frac{n(n-1)\langle T_1 T_2 (1 - T_1)(1 - T_2)\rangle}{Q_2} \\ &= \frac{(N_1 - 1)(N_2 - 1)(N_1 - 2 + \frac{2}{\beta})(N_2 - 2 + \frac{2}{\beta})}{(N - 3 + \frac{2}{\beta})(N - 4 + \frac{2}{\beta})(N - 2 + \frac{4}{\beta})},\end{aligned} \tag{9}$$

a quantity which would be important to compute the variance of the shot noise.

To be able to arrive at a more general result, we now introduce the assumption that both leads contain a large number of open channels, $N_1, N_2 \gg 1$, and thus $n \gg k$. In this case the main contribution to the summation in (5) will come from $j > k$. We can thus approximate F by

$$F \approx (\alpha + m_k)\langle P^k_{\mathbf{m}}(T)\rangle + \beta n \left\langle P^k_{\mathbf{m}}(T) \frac{T_k}{T_k - T_n} \right\rangle. \tag{10}$$

All the results obtained from now on should be understood as being valid to first order in N^{-1}. Having said that, we drop the "$\approx$" symbol and just write equalities.

We can use the identity

$$\left\langle \frac{T_k^m}{T_k - T_n} \right\rangle = \frac{1}{2}\left\langle \frac{T_k^m - T_n^m}{T_k - T_n} \right\rangle \tag{11}$$

to simplify our expression for F,

$$F = (\alpha + \beta n)\langle P^k_{\mathbf{m}}(T)\rangle + \beta \frac{n}{2}\left\langle P^{k-1}_{\mathbf{m}}(T) R^{k,n}_{m_k - 1}(T)\right\rangle, \tag{12}$$

where $R^{p,q}_a(T)$ denotes the symmetric polynomial

$$R^{p,q}_a(T) = \sum_{r=1}^{a} T_p^{a-r+1} T_q^r, \tag{13}$$

and we have neglected m_k against $\alpha + \beta n$.

Comparing (12) for m_k and $m_k - 1$ we get the relations

$$\langle P^{k-1}_{\mathbf{m}}(T) T_k\rangle = A_2 \langle P^{k-1}_{\mathbf{m}}(T)\rangle, \tag{14}$$

for $m_k = 1$ and more generally

$$\begin{aligned}\langle P^k_{\mathbf{m}}(T)\rangle &= \langle P^k_{\mathbf{m}}(T) T_k^{-1}\rangle \\ &\quad + A_1 \left\langle P^{k-1}_{\mathbf{m}}(T)[R^{k,n}_{m_k - 2}(T) - R^{k,n}_{m_k - 1}(T)]\right\rangle,\end{aligned} \tag{15}$$

for $m_k \ge 2$. In the previous equations A_1 and A_2 are the constants

$$A_1 = \frac{\beta n}{2(\alpha + \beta n)} = \frac{N_1}{N}, \quad A_2 = \frac{2\alpha + \beta n}{2(\alpha + \beta n)} = \frac{N_2}{N}. \tag{16}$$

Not surprisingly, the parameter β has dropped out of the calculation since leading-order results coincide for all universality classes. Iterating (15) k times we obtain

$$\langle P^k_{\mathbf{m}}(T)\rangle = A_2 \langle P^{k-1}_{\mathbf{m}}(T)\rangle - A_1 \left\langle P^{k-1}_{\mathbf{m}}(T) R^{k,n}_{m_k - 1}(T)\right\rangle. \tag{17}$$

Since we are interested in moments, we consider a particular case of the previous equation which is

$$\langle T_1^m\rangle = A_2 - A_1 \langle R^{1,2}_{m-1}(T)\rangle. \tag{18}$$

On the other hand, Eq. (17) also gives

$$\langle R^{1,2}_m(T)\rangle = A_2 \sum_{j=1}^{m} \langle T_1^j\rangle - A_1 \sum_{j=1}^{m-1} \left\langle T_1^{m-j} R^{2,3}_j(T)\right\rangle. \tag{19}$$

We must remark that the exponents of the terms inside the last brackets provide all ordered partitions of m into

C_{mp}	p = 1	2	3	4	5	6	7
m = 1	1						
2	1	-1					
3	1	-2	2				
4	1	-3	6	-5			
5	1	-4	12	-20	14		
6	1	-5	20	-50	70	-42	
7	1	-6	30	-100	210	-252	132

TABLE I: The values of the moment coefficients C_{mp} for several values of m and p.

3 positive integers. These two equations can now be iterated together to yield the moments $M_m = n\langle T_1^m\rangle$, which will in fact be a polynomial of degree m,

$$M_m(\xi) = N\sum_{p=1}^{m} C_{mp}\xi^p, \quad \xi = \frac{N_1N_2}{N^2}. \tag{20}$$

Finding out the coefficient C_{mp} of the power ξ^p is now an exercise in combinatorics. The first part of the problem consists in answering the following question: In how many ways can one build sequences $\{a_1, ..., a_{2p}\}$ with $a_j \in \{A_1, A_2\}$ such that both A_1 and A_2 appear exactly p times and in all subsequences $\{a_1, ..., a_q\}$, $q < 2p$ the number of A_2's is not larger than the number of A_1's. The solution to this classic problem are the celebrated *Catalan numbers*,[19]

$$c_p = \frac{1}{p+1}\binom{2p}{p}. \tag{21}$$

The power ξ^p in $M_m(\xi)$ will thus contain a factor $(-1)^{p-1}c_{p-1}$. It will also be multiplied by another factor, which is equal to the number of ordered partitions of m into p positive integers. This is $\binom{m-1}{p-1}$.

We thus obtain our main result, an explicit expression for all the moments, valid to first order in the inverse number of channels:

$$M_m(\xi) = N\sum_{p=1}^{m}\binom{m-1}{p-1}(-1)^{p-1}c_{p-1}\xi^p. \tag{22}$$

The first three moments agree with known results.[7] We present the coefficients C_{mp} with m up to 7 in Table I.

The following equation,

$$\sum_{i=1}^{n}\ln\{1+T_i[e^\lambda - 1]\} = \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}Q_k, \tag{23}$$

relates the moments and the cumulants.[2] These will also be given by polynomials, $Q_k(\xi) = N\sum_{p=1}^{k} D_{kp}\xi^p$. By feeding (23) with our result (22) we can obtain the first few cumulants, and the coefficients D_{kp} are shown in Table II. We have found by direct inspection that these coefficients are such that

$$Q_k(\xi) = N\sum_{p=1}^{k}(-1)^{k+p}\frac{(2p-2)!}{p!}S(k-1, p-1)\xi^p, \tag{24}$$

where

$$S(k,p) = \frac{1}{p!}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}j^k \tag{25}$$

are the *Stirling numbers* of the second kind.[19] From the cumulants we can derive the generating function

$$\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}Q_k(\xi) = 2N\xi\int_0^\lambda \frac{dz}{1+\sqrt{1+4\xi(e^{-z}-1)}}, \tag{26}$$

which is in fact equal to the one obtained in Ref. 7, thus implying the correctness of (24).[20]

In summary, we have explicitly obtained the random matrix theory prediction for all moments and all cumulants of the charge current in a chaotic cavity, in the limit of large channel numbers. Naturally, it would be desirable to obtain such explicit expressions for arbitrary channel numbers, but in this case we were able to compute only special cases such as (7) and (9). The moments are natural quantities to be studied in semiclassical approaches to the problem, and indeed Eq. (22) has been reproduced using action-correlated trajectories in the open quantum star graph[21]. The Hamiltonian case and corrections due to finite Ehrenfest time are discussed to some extent in Ref. 22.

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D_{kp}	p = 1	2	3	4	5	6	7
k = 1	1						
2	0	1					
3	0	-1	4				
4	0	1	-12	30			
5	0	-1	28	-180	336		
6	0	1	-60	750	-3360	5040	
7	0	-1	124	-2700	21840	-75600	95040

TABLE II: The values of the cumulant coefficients D_{kp} for several values of k and p.

[1] Ya.M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
[2] L.S. Levitov, and G.B. Lesovik, JETP Lett. **58**, 230 (1993); H. Lee, L.S. Levitov and A.Yu. Yakovets, Phys. Rev. B **51**, 4079 (1995).
[3] Yu. V. Nazarov (ed), *Quantum Noise in Mesoscopic Physics* (Dordrecht, Kluwer, 2003).
[4] B. Reulet, J. Senzier and D.E. Prober, Phys. Rev. Lett. **91**, 196601 (2003); Yu. Bomze *et al*, *ibid* **95**, 176601 (2005).
[5] C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[6] D.V. Savin and H.-J. Sommers, Phys. Rev. B **73**, 081307(R) (2006).
[7] Ya.M. Blanter, H. Schomerus and C.W.J. Beenakker, Physica E **11**, 1 (2001).
[8] O.M. Bulashenko, J. Stat. Mech. P08013 (2005).
[9] K. Richter and M. Sieber, Phys. Rev. Lett. **89**, 206801 (2002); H. Schanz, M. Puhlmann and T. Geisel, *ibid* **91**, 134101 (2003).
[10] R.S. Whitney and Ph. Jacquod, Phys. Rev. Lett. **96**, 206804 (2006).
[11] S. Heusler, S. Muller, P. Braun and F. Haake, Phys. Rev. Lett. **96**, 066804 (2006); P. Braun, S. Heusler, S. Muller and F. Haake, J. Phys. A **39**, L159 (2006).
[12] M.L. Mehta, *Random Matrices* (Academic Press, New York, 2004), 3rd edition, Chapter 17.
[13] P.G. Silvestrov, M.C. Goorden and C.W.J. Beenakker, Phys. Rev. B **67**, 241301(R) (2003).
[14] Ph. Jacquod and E.V. Sukhorukov, Phys. Rev. Lett. **92**, 116801 (2004).
[15] I. Aleiner and A. Larkin, Phys. Rev. B **54**, 14423 (1996); H. Schomerus and Ph. Jacquod, J. Phys. A **38**, 10663 (2005).
[16] O. Agam, I. Aleiner and A. Larkin, Phys. Rev. Lett. **85**, 3153 (2000); S. Oberholzer, E.V. Sukhorukov and C. Schonenberger, Nature **415**, 765 (2002).
[17] S. Rahav and P.W. Brouwer, Phys. Rev. Lett. **95**, 056806 (2005).
[18] S. Rahav and P.W. Brouwer, Phys. Rev. B **73**, 035324 (2006).
[19] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, (Cambridge, Cambridge University Press, 2001), 2nd edition.
[20] I thank H. Schomerus for pointing this out.
[21] G. Berkolaiko, J.M. Harrison and M. Novaes, unpublished.
[22] P.W. Brouwer and S. Rahav, cond-mat/0606384.